

A WEAK CHARACTERIZATION OF THE SIMPLE MAJORITY RULE

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Abstract

May's (1952) celebrated characterization of the simple majority rule appealed to four properties: decisiveness, neutrality, anonymity and positive responsiveness. The impression that they are too strong for this job was, however, extremely appealing. The aim of this paper is to present characterizations of the majority rule that appeal to properties which substantially weaken some of May's axioms. First, I prove that we can replace positive responsiveness with a weaker version of it, and still succeed in characterizing the majority rule. Secondly, I show that we can also dispense with anonymity and neutrality, and appeal to a weaker property called binary opposition.

May's (1952) celebrated characterization of the simple majority rule appealed to four properties: decisiveness, neutrality, anonymity and positive responsiveness. The impression that they are too strong for this job was, however, quite attractive. Positive responsiveness was the main subject of the attack. Some authors believed that it is not necessary, and tried to present characterizations of the majority rule that do not make use of it (Maskin: 1995; Campbell, Kelly: 2000). Others tried to use variants of it that would be more suited to uncover the structure of the defining properties of the simple majority rule (Woeginger: 2005). Anonymity was also criticized for being too strong: this is apparent in the case of infinite electorates, since it is not able to discriminate between infinite sets (Fey: 2004).

The aim of this paper is to present characterizations of the majority rule that appeal to properties which substantially weaken some of May's axioms. The primary focus is May's positive responsiveness property. I introduce two new responsiveness axioms and show that they help characterize the majority rule. They differ from May's property in that they require taking into account not two profiles of a given society, but two different societies. I prove that we can replace positive responsiveness with a weaker version of it, and still succeed in characterizing the majority rule. This new responsiveness property is shortly compared with the Maskin monotonicity. Secondly, I

show that we can also dispense with anonymity and neutrality, and appeal to a weaker property called binary opposition.

1. The framework

Let $N = \{j_1, \dots, j_n\}$ be a set of individuals, each endowed with a complete and transitive preference relation R_j on a set X of alternatives. For the purposes of this paper, it is sufficient to consider only two alternatives x and y . I write $R_j = 1$ (resp. $R_j = -1$) if the voter j strictly prefers x to y (resp. y to x), and $R_j = 0$ if j is indifferent between the two alternatives; then $R_j \in \{-1, 0, 1\}$. The preferences of the members of N are collected in a profile vector $R_N = (R_1, \dots, R_n) \in \{-1, 0, 1\}^n$. A society is a subset S of N . The preference profile of a society $S = \{i_1, \dots, i_m\} \subseteq N$ is the restriction of the preference profile R_N to S . So, the profile R_S of a society S is determined by the profile R_N of N . Since N is fixed, I shall also write simply R instead of R_N . For any profile R_S , denote by $-R_S$ the profile resulting from by R_S reversing the preferences of all the voters in it. For any two profiles $R_S = (R_1, \dots, R_m)$ and $R'_S = (R'_1, \dots, R'_m)$ we have $R_S \geq R'_S$ iff $R_i \geq R'_i$ for each i , and $R_S > R'_S$ iff $R_S \geq R'_S$ and $R'_i > R_i$ for some $i \in S$.

An aggregation rule for the society S is a function

$$F : \mathcal{C}_m \times \{-1, 0, 1\}^m \rightarrow \{-1, 0, 1\}$$

which gives the aggregate preference for any preference profile of any society.

The following three standard properties will be important in what follows.

Weak Pareto (WP): If R_S is a profile of S and $R_i = 1$ for all $i \in S$, then $F(R_S) = 1$.

Neutrality (N): $F(-R_S) = -F(R_S)$ for any profiles $R_S, -R_S \in \{-1, 0, 1\}^m$ of S .

Anonymity (A): For any profiles $R_S, R'_S \in \{-1, 0, 1\}^m$ of S , where the preferences in R_S are a permutation of the preferences in R'_S , we have $F(R_S) = F(R'_S)$.

The responsiveness axioms come in more than one variant. MR is the original May's (1952) condition. AR is its additive counterpart, explicitly introduced in Miroiu (2004). However, on some occasions it is regarded as identical to MR (see, e.g., Fey:

2004 for such a tacit supposition), although, as we shall immediately see (theorem 2) it is independent of MR. GAR is a new, and weaker property. By May's classic property MR if the society does not oppose an alternative, and a single voter in it becomes more favorable to that alternative, then the society strictly prefers it. AR states that if a society S does not oppose an alternative, and a voter who strictly prefers that alternative is added to S , then the new society follows this voter. So AR requires that new voters be taken into account and hence moving to a new society, rather than letting voters change their minds. GAR is very explicit in this sense: it states that if an indifferent society is added to a society, this does not change its aggregate preference; or, to put it differently, indifferent subgroups do not count in the aggregate preference.

May Responsiveness (MR). If $R_S > R'_S$, then $F(R_S) \geq 0$ implies $F(R'_S) = 1$. If $R'_i < R_i$, then $F(R_S) \leq 0$ implies $F(R'_S) = -1$.

Additive responsiveness (AR). Let $j \notin S$ and $S' = S \cup \{j\}$. Then for any profile R_S with $F(R_S) \geq 0$, if $R_{S'} = R_S \cup \{1\}$, we have $F(R_{S'}) = 1$. For any profile R_S with $F(R_S) \leq 0$, if $R_{S'} = R_S \cup \{-1\}$ then $F(R_{S'}) = -1$.

Group additive responsiveness (GAR). If S and S' are two societies that do not overlap (i.e. $S \cap S' = \emptyset$) and $F(R_{S'}) = 0$, then $F(R_S \cup R_{S'}) = F(R_S)$.

How are the three responsiveness properties connected? AR was used in characterizing the simple majority rule (Woeginger: 2005). I shall show that GAR is weaker than AR, and that in conjunction with A, N and WP, it is sufficient to uniquely determine the majority rule.

2. Comparing the additive responsiveness properties

Let me first introduce one more property: the null society.

Null society (NS). $F(i) = 0$.

If a society has no members, then it is indifferent on every issue. NS is not interesting if the focus is on a fixed society. In these cases the society is supposed to be non-empty, and usually to contain more than one member. But NS becomes relevant if more than one society is taken into account. We may want to define a social welfare function for each society. NS is useful when for some purposes one takes into account Boolean operations on societies; intersection is in some cases empty, while we still may want to make sense of the operation. On the other hand, when additive responsiveness axioms are at hand, it is also helpful to proceed inductively in defining a social welfare function for larger societies, the initial conditions (for a society with no members) being fixed by NS. I think it is natural to hold that the null society \emptyset neither assents nor dissents on any issue. But, if we still want to have F defined in this case, the only reasonable option is to put $F(\emptyset) = 0$.

Proposition 1. If a social welfare function satisfies N, then it satisfies NS.

Proof. For any profile vector R_S of a coalition S , let $-R_S$ be the result of inverting the preferences of the voters in S . But obviously we have $R_i = -(-R_i)$ and hence $F(R_i) = F(-(-R_i))$. By N it follows that $F(-R_i) = -F(R_i) = F(R_i) = 0$.

Proposition 2. If a social welfare function satisfies AR and N, then it satisfies WP.

Proof. By induction on the cardinality of society S we can prove that if $R_j \geq 0$ for all $j \in S$ and $R_j = 1$ for some $j \in S$, then $F(R_S) = 1$. For $n = 1$, the result yields immediately by AR and the consequence NS of N. Now assume that $n \geq 2$ and that we have already proved our proposition for $n - 1$. Let $|S| = n - 1, j \in S$ and $R_j \geq 0$. Then $|S \setminus \{j\}| = n - 1$. First, observe that for any society $S' = \{j_1, \dots, j_h\}$ if for all $j, R_{j_h} = 0$, then $F(R_{S'}) = 0$ (this requires N). There are three cases:

- 1) If $R_j = 1$, then since $F(R_S) \geq 0$, we get $F(R_S \setminus R_j) = 1$ by AR.
- 2) If $R_j = 0$, then by supposition there is some j' in S such that $R_{j'} = 1$. Let $S' = (S \setminus \{j\}) \cup \{j'\}$. Obviously, $|(S \setminus \{j\}) \cup \{j'\}| = n - 1$, and $F((R_S \setminus R_j) \cup R_{j'}) \geq 0$. By AR, $F(R_S \setminus R_j) = F(((R_S \setminus R_j) \cup R_{j'}) \setminus R_{j'}) = 1$.
- 3) If $R_j = -1$, the proof is similar to the one in case (1).

The next proposition shows that GAR does not entail AR. In fact, as we shall immediately see, GAR is weaker than AR.

Proposition 3. There is a social welfare function F that satisfies GAR, but not AR and WP.

Proof. Take F defined by: $F(R_S) = 0$ for each S . Then obviously F satisfies GAR, but not AR. Note also that F satisfies N and A. However, this function does not satisfy WP. Indeed, let $S = \{j\}$. By definition we have that $F(R_{\{j\}}) = 0$ for each profile $R_{\{j\}}$ of $\{j\}$. However, if $R_j = 1$, we must have $F(R_{\{j\}}) = 1$ whenever F satisfies WP.

Now let us introduce the simple majority rule MJ. This is the social welfare function that assigns to any profile $R_S \in \{-1, 0, 1\}^n$ of a society S the aggregate preference $MJ(R) = \text{sgn}(\sum_{j \in S} R_j)$. Here $\text{sgn}(x)$ is the standard signum-function for real numbers x defined by: $\text{sgn}(x) = 1$ for $x > 0$; $\text{sgn}(x) = 0$ for $x = 0$; and $\text{sgn}(x) = -1$ for $x < 0$.

Observe that the function F defined in the proof of Proposition 3 satisfies N, A, GAR, but is not MJ. In other words, the three properties jointly cannot characterize the simple majority rule. However, as theorem 1b in the next section proves, N, A and AR jointly succeed in characterizing MJ. Consequently, GAR is weaker than AR.

3. Weaker characterizations of the majority rule

The starting point of this section is expressed by the following two theorems:

Theorem 1. A social welfare function F is the majority rule MJ if and only if:

- a) (May: 1952) F satisfies A, N and MR; or
- b) (Woeginger: 2005) F satisfies A, N and AR.

Theorem 2. (Woeginger: 2005) Axioms R and AR are independent of each other:

- a) there is social welfare function F that satisfies MR, but not AR;
- b) there is social welfare function F that satisfies AR, but not MR.

So, in the presence of A and N, axioms MR and AR yield the same result, but they are not equivalent. An explanation of this situation is that MR as well as AR are too strong for characterizing MJ. Alternatively, the explanation might be that A and N are themselves too strong. In this section I shall investigate both lines of argument.

As with the first line, an interesting question is to see if axioms weaker than both MR and AR (as for example GAR) can be used to provide a characterization of MJ. The positive answer comes with theorem 3 below. I shall also add here proposition 4; although its main role is to help prove theorem 3, a peculiar form of it, i.e. $F(1, -1) = 0$ is very useful in many applications.

Proposition 4. If a social function F satisfies N, A and GAR, then $F(R_S) = 0$ for any profile $R_S = [k, 0, k]$.

The proposition is proved by induction on k . First, let $k = 1$. By N, we have $F(1, -1) = -F(-1, 1)$. Applying A, we get that $F(1, -1) = F(-1, 1)$, whence $F(1, -1) = -F(1, -1)$, i.e. $F(1, -1) = 0$. Secondly, suppose inductively that $F(R_S) = 0$ holds for $k - 1$. Let $R_{S'} = [k - 1, 0, k - 1]$. Then $F(R_S) = F(R_{S'} \cup \{1, -1\})$, and GAR determines that $F(R_S) = F(1, -1) = 0$.

Theorem 3 establishes the first important result of this paper. We saw that A, N and GAR do not succeed in characterizing MJ. However, adding Weak Pareto is sufficient to reach the desired result.

Theorem 3. A social welfare function F satisfies A, N, WP and GAR if and only if it is the simple majority rule MJ.

Proof. Since majority rule clearly satisfies the four axioms, necessity is straightforward. For the sufficiency part, let an arbitrary social welfare function F satisfy axioms A, N, WP and GAR. Since F satisfies A, we may represent an n -voter profile R_S of a society S by a triple $[p, z, m]$ (plus, zero, minus) of non-negative integers, such that p of the voters have $R_i = 1$, z of the voters have $R_i = 0$, and m of the voters have $R_i = -1$. The idea of the proof is to discount first the group of indifferent voters. Then groups formed of an equal number of for and against voters are isolated. Finally, we are left with a

society whose members all vote either for, or against an alternative, and WP yields immediately the desired result.

So, let $S_z \subseteq S$ be the set of all voters j for which $R_j = 0$. Now, by N it follows that $F(S_z) = 0$, and with GAR we get $F(R_S) = F(R_{S-S_z})$.

- 1) If $m = 0$ and $p = 0$, then $S - S_z = i$ and by proposition 1 we have that $F(R_{S-S_z}) = 0 = F(R_S) = MJ(R_S)$.
- 2) If $m = 0$ and $p > 0$, then $F(R_{S-S_z}) = 1$ by WP, and thus $F(R_S) = 1 = MJ(R_S)$.
- 3) The case when $p = 0$ and $m > 0$ is symmetric to the second case.
- 4) Let $m > 0$, $p > 0$ and $p > m$. Since $p > m$, there is a set $S' \subseteq S - S_z$ such that $R_{S'} = [m, 0, m]$. By proposition 4, $F(R_{S'}) = 0$. Then by applying GAR we get that $F(R_{S' \subseteq S-S_z-S'}) = F(R_{S-S_z-S'})$. But $R_{S-S_z-S'} = [p-m, 0, 0]$, and by hypothesis $p-m > 0$. Then WP yields $F(R_{S-S_z-S'}) = 1 = F(R_{S-S_z}) = F(R_S) = MJ(R_S)$.
- 5) Finally, the case when $m > 0$, $p > 0$ and $m > p$ is symmetric to the fourth case.

Our property GAR can be nicely compared with the well-known monotonicity condition introduced in Maskin (1999). Maskin monotonicity is defined by: if R_S and R'_S are two profiles of S , then: (i) if $R_i \geq 0$ entails $R'_i \geq 0$ for each $i \in S$, then $F(R_S) \geq 0$ entails $F(R'_S) \geq 0$; (ii) if $R_i \leq 0$ entails $R'_i \leq 0$ for each $i \in S$, then $F(R_S) \leq 0$ entails $F(R'_S) \leq 0$. It has been proved that MM can be used to characterize absolute q -majority rules. Let $q \in \{m^*, \dots, m\}$, where m^* be the lowest integer exceeding $m/2$. Then a q -majority rule is a social welfare function such that for each profile R_S of S we have that $F(R_S) = 1$ iff $|\{i \in S: R_i = 1\}| \geq q$, and $F(R_S) = -1$ iff $|\{i \in S: R_i = -1\}| \geq q$. Aşan and Sanver (2006) showed that:

Proposition 5. A social welfare function F satisfies A, N, WP and MM if and only if F is an absolute q -majority rule for some $q \in \{m^*, \dots, m\}$.

Now we may easily notice by comparing this proposition with theorem 3 that adding GAR to A, N and WP we get MJ; while adding MM to them we get an absolute q -majority rule. However, the simple majority rule MJ does not satisfy MM and hence it is

not an absolute q -majority rule. The following proposition expresses more generally the way in which GAR and MM diverge.

Proposition 6. Let WP and N hold for F . Then F cannot satisfy both MM and GAR.

Proof. Under WP and N, assume that F satisfies GAR. I shall prove that it does not satisfy MM. To do this, observe first that F also satisfies SP. Let S_0 be the set of all the voters in S who are indifferent. Then the societies $S - S_0$ and S_0 are disjoint. By N, we have that $F(R_{S_0}) = 0$, and thus $F(R_S) = F(R_{S - S_0})$. Suppose that the society $S - S_0$ is not empty and that $R_i = 1$ for each i in it. Then by WP we get that $F(R_{S - S_0}) = 1$, and GAR entails that $F(R_S) = 1$, i.e. F satisfies SP. On the other hand, it is known that if a social welfare function F satisfies MM, then it does not satisfy SP. By contraposition, if F satisfies SP, it does not satisfy MM. Consequently, F does not satisfy MM.

Alternatively, we may try to characterize MJ by dispensing with neutrality and anonymity, and make use of weaker properties. Consider the property of binary opposition:

Binary opposition (BO). If $R_i = -R_j$, then $F(R_{\{i,j\}}) = 0$.

We saw in the proof of Proposition 4 that BO is entailed by N and A when the two voters have opposing preferences; and N is enough for this when both voters are indifferent. On the other hand, clearly BO is weaker than both N and A.

Binary opposition is strong enough to yield the results we need in the proof of the characterization theorem 4. By BO, we have that $F(1, -1) = 0$. Moreover, BO entails that $F(0, 0) = 0$. Third, let $i = j$ and $R_i = 0$; then the condition in the antecedent of BO is satisfied, whence we get that $F(R_{\{i\}}) = 0$.

Theorem 4. A social welfare function F satisfies BO, WP and GAR if and only if it is the simple majority rule MJ.

The proof replicates the procedure described in the proof of Theorem 3. By BO, we can first eliminate the indifferent voters (GAR is also needed). Then we appeal again to BO and GAR to eliminate couples of voters who have opposing preferences. Finally, WP yields the group decision.

The characterization of the simple majority rule MJ which is rendered by Theorem 4 appeals to quite weak properties. First, GAR is weaker than AR. Secondly, the characterization avoids neutrality N by relying on the weaker property BO. Finally, anonymity is dispensed with. Its job is done jointly by GAR and BO. They appeal to sets of voters: their order is not significant, and we are required to take into account only the way in which they vote.

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