POSITIVE RESPONSIVENESS AND INFINITE ELECTORATES

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0. **Introduction**

In a highly influential paper K. May [10] characterized the simple majority rule on a finite group of voters in terms of three properties of social choice functions: neutrality, anonymity and responsiveness. However, attempts to generalize May's result to infinite sets of voters are not smooth. In a number of cases the appeal to infinite populations is significant ([5], [7]). Infinite societies may represent for example future generations, or finite societies with members extending into the indefinite future, or political parties competing in an indefinitely long sequence of elections etc. Infinite sets bring about difficulties one does not encounter in the finite case. Size is an example: all countable infinite sets are in one sense of the same 'size', meaning that there is a one-to-one map between any two of them. As one may expect, size raises the question of interpreting May's anonymity condition on infinite domains ([11], [5, [4]).

In this context, May's responsiveness property did not look to play a significant role. While many authors rejected responsiveness, and tried to present characterizations of the majority rule that do not appeal to it ([9], [3]), responsiveness was not overtly involved in set-theoretical arguments. In this paper I argue that when the electorate is infinite, the set-theoretical Axiom of Choice (**AC**) has a direct bearing on the May-type responsiveness property. I show that part of the import of responsiveness (in cases in which infinite collections of voters are allowed) is that **AC** holds. Roughly, **AC** states that out of any collection of non-empty sets we can produce a new set by selecting exactly one element of each set (Levy: [8]). It is known that AC is equivalent to the so-called Zorn's Lemma: the claim that each set can be well-ordered. Zorn's Lemma is

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required in the proof that free ultra-filters on infinite sets exists. Fey [4] argues that the appeal to free ultra-filters is pivotal for a correct generalization of the anonymity condition in the infinite case. The role of **AC** was also investigated in Brunner and Mihara [2] in the context of Arrow-type social welfare functions defined for infinite societies.

The plan of the paper is this. In the first section I introduce the framework of analysis. In the second one coalition structures are defined and some of the properties of monotonic ones are studied. In section 3 I give a new formulation of responsiveness, and show that this appeals directly to **AC**.

1. **The framework**

Let $N = \{j', j'', j'''\ldots\}$ be a group of voters. For the moment I shall make no assumption on the cardinality of N: it can be either finite or (countably) infinite. Let X be the collection of alternatives the members of the group N face. For the purposes of this paper, it is sufficient to consider only two alternatives, x and y. I assume that the preferences of the voters are linear orders. A profile on N is a function $\mathbf{p}: N \rightarrow \{1, -1\}$. If $\mathbf{p}(j) = 1$, then j strictly prefers x to y; if $\mathbf{p}(j) = -1$, then j strictly prefers y to x. A coalition is a subset of N. Coalitions will be denoted by A, B etc. For each coalition A, its complement (relative to N) is denoted by A^{C}. A permutation on N is a mapping $\pi: N \rightarrow N$ which is one-to-one and onto; $\pi(A) = \{j' \ 0\ N$: there is some $j 0\ A$ such that $\pi(j) = j'\}$ denotes the image set of A under the permutation π.

For each profile $\mathbf{p}$, an aggregation rule is a function $F_{\mathbf{p}}$: $(N) \rightarrow \{1, 0, -1\}$, which gives the aggregate preference for any preference profile of any coalition. Here (N) is the power set (the set of all subsets) of N and the value 0 is interpreted as a tie. Although voters are not indifferent, social indifference is here allowed.

Well known properties of aggregation functions are:

Weak Pareto (**WP**): $F_{\mathbf{p}}(N) = 1$.

Neutrality (**Neu**): $F_{\mathbf{p}}(A^{C}) = -F_{\mathbf{p}}(A)$

Anonymity (**A**). $F_{\mathbf{p}}(A) = F_{\mathbf{p}}(\pi(A))$.

Monotonicity (**Mon**): For all A and B, if $A \ f\ B$, then $F_{\mathbf{p}}(A) \leq F_{\mathbf{p}}(B)$.

Additive Responsiveness (**AR**): If A B and $F_{\mathbf{p}}(A) \geq 0$, then $F_{\mathbf{p}}(B) = 1$.[2]

Since for the rest of this paper the profile **p** is kept constant, I shall write simply F instead of $F_{\mathbf{p}}$.

A core notion I shall use is that of a coalition structure. A coalition structure is a collection **W** f   (N) of coalitions such that for no A f N both A 0 **W** and A^C 0 **W** hold.[3] The coalition structure **W** is called monotonic if B0 **W** whenever A0 **W** and A f B[4]. Coalitions in **W** are called winning. A 0 **W** is a minimal winning coalition if for all B A it is false that B 0 **W**. A coalition A f N is almost winning if for all B such that A B we have that: B0 **W**. (Clearly, all winning coalitions are almost winning.)

2. **Coalition structures and aggregation functions**

Coalition structures and aggregation functions can be correlated in a systematic manner, as the next propositions show. The idea is to represent the properties of aggregation functions as properties of coalition structures.

Proposition 1.

(i) Each neutral aggregation rule determines a coalition structure.

(ii) Each coalition structure determines a neutral aggregation rule.

Proof. Let F be neutral. The coalition structure $\mathbf{W}_F$ is constructed as follows: $\mathbf{W}_F$ = {Af N: $F(A) = 1$}. Observe that by neutrality, if $F(A) = -1$, then $F(A^C) = 1$ and therefore A^C 0 $\mathbf{W}_F$; and if $F(A) = 0$ (and so A ó $\mathbf{W}_F$ by definition), then $F(A^C) = 0$ and therefore A^C ó $\mathbf{W}_F$. Moreover, since F is a function, we shall never have both A 0 $\mathbf{W}_F$ and A^C 0 $\mathbf{W}_F$. Conversely, let **W** be a coalition structure. Then we may construct an aggregation rule $F_{\mathbf{W}}$ as follows: $F_{\mathbf{W}}(A) = 1$ if A 0 **W**; $F_{\mathbf{W}}(A) = -1$ if A^C 0 **W**; and $F_{\mathbf{W}}(A) = 0$ if neither A 0

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[2] Here A B means that A f B and B é A.

[3] Some authors give stronger definitions of a coalition structure. For example, Barbera, Massó and Neme [1] define a coalition structure as a nonempty family of nonempty coalitions of N, which satisfies coalition monotonicity.

[4] A monotonic coalition structure satisfying:

If A 0 **W** and B 0 **W**, then A 1 B 0 **W**

is a filter. We can easily see that this condition corresponds to the following property of an aggregation function F:

Distributivity (**D**): If $F(A) = 1$ and $F(B) = 1$ and A 1 $B \neq$ i, then $F(A$ 1 $B) = 1$.

The majority rule violates distributivity; the extended Pareto rule satisfies it.

W nor A^C 0 **W**. Since **W** does not include both a coalition and its complement, the function $F_{\mathbf{W}}$ is well defined. Obviously, $F_{\mathbf{W}}$ is neutral. Note that if $F(A) = 0$ for all A, then **W** is empty. I shall assume, however, that **W** contains at least one element, i.e. $F(A) \neq 0$ for some A.

Coalition structures like **W** have an intuitive interpretation. For example, in the case of the simple majority rule, the members of **W** represent majorities of individuals. A choice A is selected according to this rule, i.e. $F(A) = 1$, if the voters in it form a majority. In the case of an absolute q-majority rule the members of **W** are collections of individuals with at least q members, where $q \geq n^*$ (the lowest integer exceeding $|N|/2$) etc.

The properties of aggregation rules can be easily translated into properties of the coalition structures.

Proposition 2. Let F be neutral. Then at $\mathbf{W}_F$ properties **WP**, **A**, **Mon** and **AR** correspond, respectively, to:

($\mathbf{WP_W}$) N 0 $\mathbf{W}_F$.

($\mathbf{A_W}$) A 0 $\mathbf{W}_F$ iff $\pi(A)$ 0 $\mathbf{W}_F$.

($\mathbf{Mon_W}$) If A f B and A 0 $\mathbf{W}_F$, then B 0 $\mathbf{W}_F$.

($\mathbf{AR_W}$) If A B and A^C ó $\mathbf{W}_F$, then B 0 $\mathbf{W}_F$.

The anonymity property here defined is a very strong one, because it puts no constraint on permutations π. A weaker form is finite anonymity. A permutation π is finite if there is an integer m such that $n > m$ implies $\pi(n) = n$; and a social welfare function F satisfies finite anonymity if for all A and all finite permutations π, $F(A) = F(\pi(A))$. The property of coalitions structures corresponding to finite anonymity is:

($\mathbf{FA_W}$) A 0 $\mathbf{W}_F$ iff $\pi(A)$ 0 $\mathbf{W}_F$ for all finite permutations π.

Now let us consider monotonicity. The correlation can be established in this case in a weaker manner, as proposition 3 shows. This correlation will be essential in the next section.

Proposition 3.

(i) If **W** is a monotonic coalition structure, then $F_{\mathbf{W}}(A) = 1$ iff there is some C such that C0 $\mathbf{W}_F$ and C f A.

(ii) If F is neutral and monotonic, then A 0 $\mathbf{W}_F$ iff there is some C such that C0 $\mathbf{W}_F$ and C f A.

Proof. I shall give only the proof of (ii). Again, let $\mathbf{W}_F = \{A \subseteq N: F(A) = 1\}$. Since F is monotonic, if $C \subseteq A$ and $C \in \mathbf{W}_F$, then $A \in \mathbf{W}_F$. First, suppose that $A \in \mathbf{W}_F$. then there is some C, i.e. exactly A, such that $C \in \mathbf{W}_F$ and $C \subseteq A$. On the other hand, suppose that there is some C such that $C \in \mathbf{W}_F$ and $C \subseteq A$. Then $F(C) = 1$, and by monotonicity we have $F(A) = 1$, and thus $A \in \mathbf{W}_F$.

The following corollary will be used directly:

Proposition 4. If **W** is a monotonic coalition structure, then $F_{\mathbf{W}}(A) \geq 0$ iff $A \cap C \neq \emptyset$ for each $C \in \mathbf{W}$.

Proof. Observe that $F_{\mathbf{W}}(A) = -1$ iff there is some $C \in \mathbf{W}$ such that $C \subseteq A^C$. Then $F_{\mathbf{W}}(A) \geq 0$ iff it is not true that there is some $C \in \mathbf{W}$ such that $C \subseteq A^C$, or equivalently: for all $C \in \mathbf{W}$ it is false that $C \subseteq A^C$, i.e. $A \cap C \neq \emptyset$.

Proposition 5. If F is monotonic and neutral, then all the coalitions in the coalition structure $\mathbf{W}_F$ are non-empty.

Proof. Suppose that $\mathbf{W}_F$ is non-empty. By monotonicity we have that $N \in \mathbf{W}_F$. Since by neutrality $\mathbf{W}_F$ does not include both a coalition and its complement it follows that $\emptyset \notin \mathbf{W}_F$. (Note: Proposition 5 does not exclude the possibility that $\mathbf{W}_F$ be itself empty.)

3. **The Responsiveness property**

By proposition 2, for a neutral function F, **AR** expresses a quite simple condition on the structure of the coalition structures: $\mathbf{AR}_{\mathbf{W}}$. But something subtler happens if monotonicity is added. By proposition 3 for each monotonic coalition structure **W** we can generate a neutral and monotonic aggregation function $F_{\mathbf{W}}$ (or F for short). Let $F(A) \geq 0$ for some A. Since **W** satisfies monotonicity, we have that $A \cap C \neq \emptyset$ for each $C \in \mathbf{W}$. Therefore $\mathbf{W}(A) = \{A \cap C : C \in \mathbf{W}\}$ contains no empty members.

A choice function for a set **H** of non-empty sets is a function $\chi: \mathbf{H} \rightarrow \cup\mathbf{H}$ such that $\chi(D) \in D$ for each $D \in \mathbf{H}$. The set-theoretical Axiom of Choice (**AC**) states that functions χ exists for each **H** (Levy [8]). Given a set **H**, $\chi(\mathbf{H})$ is a choice set. When **H** is $\mathbf{W}(A)$[5], we

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[5] Observe that since **W** is monotonic, by proposition 5 all its members are nonempty, and thus the function χ is well defined.

can define a choice function χ by: for each C0**W**, $\chi(C)$ 0 A 1 C, and let $\chi(\mathbf{W}(A))$ be the collection of all voters we pick up in this way. Obviously, for each A we have that $\chi(\mathbf{W}(A))$ f A.

The main result of this section is expressed by the following proposition:

Proposition 6. Suppose that **W** is a monotonic coalition structure. Now F satisfies **AR** iff for all A such that $F(A) \geq 0$, $\chi(\mathbf{W}(A))$ is an almost winning coalition.

Proof. First observe that if N is infinite, then for each A f N the Axiom of Choice **AC** guarantees that the set $\chi(\mathbf{W}(A))$ exists. For the 'if' part, suppose that for all A such that $F(A) \geq 0$, $\chi(\mathbf{W}(A))$ is an almost winning coalition, and that F does not satisfy **AR**. Since F does not satisfy **AR**, we have for some A and B that A B (i.e. A f B and $B - A \neq$ i) and $F(A) \geq 0$, but $F(B) \leq 0$. $F(A) \geq 0$ entails that $\chi(\mathbf{W}(A))$ is a choice set, and by supposition it is an almost winning coalition. Specifically, we have that $\chi(\mathbf{W}(A))$ c $(B - A)$ is a winning coalition. By $F(B) \leq 0$, for each C 0 **W** there is some j0C such that j ó B and also jóA (because of A f B). But this condition fails for $C = \chi(\mathbf{W}(A))$ c $(B - A)$ 0 **W**, because for all j 0 $\chi(\mathbf{W}(A))$ c $(B - A)$ we also have j 0 B. For the 'only if' part, suppose that **AR** holds, and $F(A) \geq 0$. We have that if $F(A) \geq 0$, then $F(\chi(\mathbf{W}(A))) \geq 0$. For since $F(A) \geq 0$, the set $\chi(\mathbf{W}(A))$ is well-defined. Then by the definition of χ we have that $\chi(\mathbf{W}(A))$ 1 $C \neq$ i for each C0 **W**. Proposition (4) entails that $F(\chi(\mathbf{W}(A))) \geq 0$. Finally, by **AR** we get that $\chi(\mathbf{W}(A))$ is almost winning.

If the set N of voters is infinite, then coalition structures built on it can contain an infinite number of coalitions. For example, if **W** contains the set of even integers, then by monotonicity all coalitions including it are members of the **W**. But then the existence of $\chi(\mathbf{W}(A))$ requires the axiom of choice **AC**. Proposition 6 shows that, on infinite monotonic coalition structures, responsiveness requires that **AC** be assumed. This result is significant in that it appeals directly to **AC**, and not to its equivalent: Zorn's lemma.[6]

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[6] Generalizing May's theorem to infinite electorates requires to interpret the anonymity condition. As Fey [4] showed, the property **A** is too strong because it cannot discriminate between infinite sets. Conversely, finite anonymity is too weak because it permits arbitrarily small minorities to overrule large majorities. Fey appealed to an intermediate property (bounded anonymity), which requires invariance to similarly sized sets, as measured by an asymptotic density measure d. He introduced a density responsiveness property (**DR**): a function F satisfies **DR** if for all A and B, $F(A) \geq 0$, A B and $d(B - A) > 0$ entail that $F(B) = 1$. He proved that if an aggregation rule F satisfies neutrality, density responsiveness and bounded anonymity, then it agrees with density

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majority rule. An interesting topic is then to apply the notion of a density measure to winning coalitions. We can advance the following definition of a winning coalition: a coalition A is almost density winning if for all B such that A B and $d(B - A) > 0$ we have that: B 0 **W**.