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WORLD-INDEXED SENTENCES AND MODALITY

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Abstract
The paper investigates propositional modal languages which contain indexes referring to possible worlds. A basic possible worlds logic is defined, in which we can mimic standard conditions in the usual modal semantics on the behavior of possible worlds. Connections between modality and world-indexed sentences are then studied. A semantics in terms of 'possible worlds' is defined, and completeness results relative to special classes of models called 'mirror models' are proved.

A world-indexed sentence has the form: 'In world w, Quine is a distinguished philosopher'. Interesting cases involve iteration, for example: 'In world w' it is the case that in world w Quine is a distinguished philosopher'. I will study in this paper the logic of these sentences. Intuitively, we may try to construct a language in which such sentences are allowed, and in which we can mimic standard conditions in the usual modal semantics on the behavior of possible worlds.

Section 1 is devoted to the investigation of this basic logic of the possible worlds. Section 2 introduces modality and investigates the connections between modal and world-indexed sentences. A semantics in terms of 'possible worlds' for this new logic is defined. In section 3 a new modal logic of possible worlds is introduced, and its completeness relative to special classes of models called 'mirror models' is proved.

1. *Basic possible worlds logic*

Let the language $\mathcal{I}$ consists of a denumerable set S of propositional letters S, S′, S″ etc.; we also assume that it contains a set W of (possible) worlds terms w, w', w'' etc. The sentences of $\mathcal{I}$ are the members of the smallest set that includes:

(i) every propositional letter in S;
(ii) $\varphi \vee \psi$, whenever φ and ψ are in it;
(iii) $\neg\varphi$ and $w\varphi$, whenever φ is in it, and $w \in W$[1].

The following conditions on the behavior of the world-indexed sentences seem intuitively plausible:

1.1. $\neg w\varphi \equiv w\neg\varphi$
1.2. $w(\varphi \vee \psi) \equiv (w\varphi \vee w\psi)$
1.3. If $\vdash \varphi$, then $\vdash w\varphi$, for each $w \in W$.

BW (basic possible worlds logic) has as axioms all tautologies, all expressions of the form (1.1), (1.2) and is closed under the rules of detachment and (1.3). In BW the concept of reflection of a world by another is central, as we shall immediately see. Observe that (1.1)–(1.3) mimic in our language $\mathcal{I}$ standard conditions on the behavior of possible worlds: each world is consistent (a sentence and its negation cannot be both true in any world); and every possible world is maximal (for every sentence, either it or it negation holds at it). Indeed, consistency is expressed by (1.1). Further, from the tautology

1.4.1. $\varphi \vee \neg\varphi$

we get:

1.4.2. $w(\varphi \vee \neg\varphi)$ (by 1.3)
1.4.3. $w\varphi \vee w\neg\varphi$ (by 1.2)

and hence w is maximal. If the other logical connectives are defined as usual, the following results yield easily:

1.5.1. $w(\varphi \wedge \psi) \equiv (w\varphi \wedge w\psi)$
1.5.2. $w(\varphi \rightarrow \psi) \equiv (w\varphi \rightarrow w\psi)$
1.5.3. $w(\varphi \equiv \psi) \equiv (w\varphi \equiv w\psi)$

Moving to semantics, a model of BW is a triple $\mathsf{M} = \langle \Pi, \models, F \rangle$ where Π is a collection of 'possible worlds' τ, τ', τ'', etc., F is a function $(W \times \Pi) \rightarrowtail \Pi$, and relation $\models$ is defined by:

[1] The case is in fact more complicated. For each $w \in W$, we should first define an operator $\langle w \rangle$, with the meaning: in the world w... Then, if φ is a sentence, $\langle w \rangle\varphi$ will be a sentence too. However, since w and $\langle w \rangle$ are one-to-one correlated, I will ignore this complication.

1.6.1. If $\varphi \in S$, then $\tau \models \varphi$, or $\tau \models \neg\varphi$.
1.6.2. $\tau \models \neg\varphi$ iff it is not the case that $\tau \models \varphi$.
1.6.3. $\tau \models \varphi \vee \psi$ iff $\tau \models \varphi$ or $\tau \models \psi$.
1.6.4. $\tau \models w\varphi$ iff $F(w, \tau) \models \varphi$.

A sentence φ is true at M, and I shall and write $\mathsf{M} \models \varphi$ for this, iff for every $\tau \in \Pi$, $\tau \models \varphi$; and it is BW-valid iff it is true at every model of BW. I shall write $\models_{\mathrm{BW}} \varphi$ in this case. (However, whenever there is no danger of confusion, I will omit the index.) The intuition behind the definition (1.6.4) is simple: each world in our W-collection is correlated, relative to each τ, with a 'possible world' $F(t, \tau) = \tau'$ from our Π-collection. Whenever a sentence φ is true at τ', the sentence $w\varphi$, i.e. the sentence that φ is the case at w, is also true at τ. As seen from τ, the 'possible world' τ' looks like w. Or, to put it in another way, τ' is reflected or mirrored in τ like w. The concept of reflection of a world within another world is central in my approach to world-indexed sentences. I discussed it to a larger extent in Miroiu (1997, 1999).

Notice that the two collections W and Π of worlds must be kept distinct. We have yet no reason to take their members as been identical, or at least to assume that they are systematically correlated. The only thing we have by now is that at each 'world' τ every world w reflects (*via* function F) a 'world' τ'. But we are not justified to claim that w is exactly τ'.

Our language $\mathcal{I}$ allows for iteration of the world operators; not only $w\varphi$, but also $ww'\varphi$ and $ww'w''\varphi$, etc. are sentences. We may state that φ is the case at w, but also that at w it is the case that at w' it is the case that φ, etc. Now one can easily check that in our semantical framework there is no logical connection between $\tau \models ww'\varphi$ and $\tau \models w\varphi$ or $\tau \models w'\varphi$.[2]

[2] This runs counter the common view on indexed sentences. According to it, if $\tau \models w\varphi$, then for any w' it also true that $\tau \models ww'\varphi$. If φ is the case at w, then from the point of view of any other world w' this is an unalterable fact: that φ is the case at w is bound to be the case at every world w'. If Quine happens to be a distinguished philosopher in the actual world, then for any other world w Quine is bound to be actually a distinguished philosopher. A similar argument runs for so called world-indexed properties. As A. Plantinga put it, 'we say that a property P is world-indexed if there is a world W and a property Q such that P is equivalent to the property of having Q in W or to its complement — the property of not having Q in W... But an interesting peculiarity of world-indexed properties, as we have seen, is that nothing in any world has any such property accidentally.' (Plantinga: 1970, pp. 490–2)

In our frame the common view on indexed sentences can be obtained as a special case. We may directly add to our logic BW a new axiom:

(1) $w\varphi \equiv w'w\varphi$
According to axiom (1) the iteration of world operators is superfluous. Semantically, it defines the following property of the function F:

(2) $F(w, \tau) = F(w', F(w, \tau))$
Indeed, given (2) we have: $\tau \models w\varphi$ iff $F(w, \tau) \models \varphi$ iff $F(w', F(w, \tau)) \models \varphi$ iff $F(w, \tau) \models w'\varphi$ iff $\tau \models ww'\varphi$.

Let us mention an important theorem of BW. Starting from (1.4.3) we get:

1.7.1. $w'(w\varphi \vee w\neg\varphi)$
1.7.2. $w'w\varphi \vee w'w\neg\varphi$

According to (1.7.2) at world w' either $w\varphi$ is the case, or $w\neg\varphi$ is the case: so, world w' mirrors w, i.e. it creates within itself a complete image of what is going on at w, in that for every φ, at w' it is the case that either φ is the case at w, or $\neg\varphi$ is the case at w. (Again, there is no guarantee that the image of w in w' is adequate, i.e. that world w is mirrored in w' as it 'really' is.) We can express this idea more rigorously by means of the following lemma:

1.8. (*The mirroring lemma*) If Σ is a BW-maximal consistent set of sentences of $\mathcal{I}$, then $\Sigma_w = \{\varphi : w\varphi \in \Sigma\}$, with $w \in W$, is BW-maximal consistent.

Proof. Σ_w is BW-consistent. For suppose it is not. Then for some ψ we have both $\psi \in \Sigma_w$ and $\neg\psi \in \Sigma_w$. But, according to the definition of Σ_w, both $w\psi \in \Sigma$ and $w\neg\psi \in \Sigma$ must hold. By (1.1), we get $\neg w\psi \in \Sigma$, in contradiction with our assumption that Σ is BW-maximal consistent. Secondly, Σ_w is BW-maximal. Suppose it is not so. Then for some ψ neither $\psi \in \Sigma_w$ nor $\neg\psi \in \Sigma_w$. But (1.4.3) is a theorem of BW, hence $w\psi \vee w\neg\psi \in \Sigma$ ans so either $w\psi \in \Sigma$, or $w\neg\psi \in \Sigma$. But in this case we have either $\psi \in \Sigma_w$, or $\neg\psi \in \Sigma_w$ — contradiction.

Now let Σ be a BW-maximal consistent set of sentences of $\mathcal{I}$. Then Σ must contain complete descriptions Σ_w of what happens at each world w. Moreover, (since it contains maximal sets like $\Sigma_{ww'}$) Σ_w must contain complete descriptions of the way in which the world w mirrors what happens in any world w'. But, of course, we have yet no reason to suppose that what Σ_w claims that is going on at w' is identical with what $\Sigma_{w''}$ claims that is going on at w', or with what is 'really' going on at world w', i.e. with the way $\Sigma_{w'}$ really is. More formally, from $\tau \models ww'\varphi$ we cannot infer anything about $\tau \models w''w'\varphi$ or $\tau \models w'\varphi$. I will return to this issue in the next section. The present ends with an expected completeness result:

1.9. $\vdash_{\mathrm{BW}} \varphi$ iff $\models_{\mathrm{BW}} \varphi$.

Proof. Sufficiency is immediate. To prove (1.1), let $\mathsf{M} = \langle \Pi, \models, F \rangle$ be a model of BW, and let $\tau \in \Pi$. Then: $\tau \models \neg w\varphi$ iff it is not the case that $\tau \models w\varphi$, iff it is not the case that $F(w,\tau) \models \varphi$, iff $F(w,\tau) \models \neg\varphi$, iff

$\tau \models w\neg\varphi$. To prove (1.2), we have: $\tau \models w(\varphi \vee \psi)$ iff $F(w,\tau) \models \varphi \vee \psi$, iff $F(w,\tau) \models \varphi$ or $F(w,\tau) \models \psi$, iff $\tau \models w\varphi$ or $\tau \models w\psi$, iff $\tau \models w\varphi \vee w\psi$. Further, suppose that (1.3) is not valid. Then $\tau \models \varphi$ for all τ, but $\tau' \models w\varphi$ is not the case for some τ'. Thus, it is not the case that $F(w,\tau') \models \varphi$. But $F(w,\tau') = \tau''$, and so it is not the case that $\tau'' \models \varphi$ for some τ'' — contradiction.

To prove the necessity part of (1.9), suppose that φ is not BW-provable. Then there is some model $\mathsf{M} = \langle \Pi, \models, F \rangle$ such that for some $\tau \in \Pi$, $\tau \models \varphi$ is not the case. We will construct this model as follows: Π is the set of all BW-maximal consistent set of sentences of $\mathcal{I}$. If Σ is in Π, put $F(w,\Sigma) = \Sigma_w$. By (1.8), we know that Σ_w is also BW-maximal consistent, and hence $\Sigma_w \in \Pi$. Further, put $\Sigma \models \varphi$ iff $\varphi \in \Sigma$, whenever φ is a sentence S in S. To show that M is a model of BW we need to prove that for all φ, $\Sigma \models \varphi$ iff $\varphi \in \Sigma$. The only difficult case is when $\varphi = w\psi$, for some ψ. We have: $\Sigma \models w\psi$ iff $F(w,\Sigma) = \Sigma_w \models \psi$, iff $\psi \in \Sigma_w$, iff $w\psi \in \Sigma$. Indeed, $\psi \in \Sigma_w$ entails $w\psi \in \Sigma$. For suppose it did not. Then, according to the definition of F and the mirroring lemma, ψ were not in Σ_w. The converse results by an application of the definition of F. Finally, since φ is not a theorem of BW, $\{\neg\varphi\}$ is BW-consistent. It can then be extended to a BW-maximal consistent set Σ, and $\neg\varphi \in \Sigma$, i.e. it is not the case that $\Sigma \models \varphi$. But Σ is a member of the set Π, which means that for some Σ, φ is not true in Σ, q.e.d.

2. *Modality and possible worlds*

Let us enrich our language with a new unary operator $\Box$. We get thus a new language $\mathcal{I}'$. Intuitively, $\Box\varphi$ means that φ is necessary. In the usual manner, starting from the necessity operator, the possibility operator $\Diamond$ can be defined: $\Diamond\varphi =_{\text{df}} \neg\Box\neg\varphi$. The aim of this section is to build a logic in which modal sentences like $\Box\varphi$ and $\Diamond\varphi$ mix with indexed sentences. I shall argue that modal operators are very useful in the attempt to study the logic of world-indexed sentences. This is the main reason why the logical properties of the modal operators will be investigated. To start with, let us add to BW two new axioms and two new rules[3]. We get a new logic BWM (basic modal possible worlds logic).

2.1. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
2.2. $\Box\varphi \to w\varphi$
2.3. If $\vdash \varphi$, then $\vdash \Box\varphi$

[3] Almost all concepts and results used without proof in this paper can be found in Bull and Segerberg's (2001) short introduction to modal logic. See also Blackburn, de Rijke, Venema (2001).

2.4. If $\vdash w\varphi$ for all w, then $\vdash \Box\varphi$

(2.4) is a kind of ω-rule. If the set W of names for possible worlds in our language $\mathcal{I}'$ is infinite, then this rule is not necessarily effective, i.e. we cannot be sure that the collection of the theorems of our logic is recursively enumerable. The cardinality of the set W is also important in semantics. In the next section I shall prove that the completeness theorem for some modal logics of possible worlds can be established by use of models the cardinality of which is exactly $\mid W \mid$.

A model for BWM is again a structure $\mathsf{M} = \langle \Pi, \models, F \rangle$. Here Π and F are defined as above. The definition of $\models$ must be extended to cases in which the operator $\Box$ is involved. Intuitively, a sentence φ is necessary at some world τ if and only if at τ the sentence that φ is the case at w is true for every world w:

1.6.5. $\tau \models \Box\varphi$ iff $\tau \models w\varphi$ for every w.

The definition of a sentence $\Box\varphi'$ being true at a world $\tau \in W$ appears to be different from the usual treatment of the operator $\Box$. First, a sentence like $\Box\varphi$ is given a greater complexity as for example $w\varphi$. Indeed, according to (1.6.5) the definition of $\tau \models$ for a modal sentence presupposes the definition of $\tau \models$ for world-indexed sentences. Secondly, the usual strategy of defining $\tau \models \Box\varphi$ is to use an accessibility relation $\mathsf{R}(\tau', \tau)$ between possible worlds and have something like:

1.6.6. $\tau \models \Box\varphi$ iff $\tau \models \varphi$ for all τ' such that $\mathsf{R}(\tau', \tau)$.

Observe, however, that we can appeal to simple trick to get something very similar to (1.6.6). The following four expressions are equivalent (to shorten the argument, I will put them in a more formal dress):

(2.5.1) $\tau \models \Box\varphi \equiv (\forall w)\tau \models w\varphi$
(2.5.2) $\tau \models \Box\varphi \equiv (\forall w)F(w, \tau) \models \varphi$
(2.5.3) $\tau \models \Box\varphi \equiv ((\forall \tau')(\forall w)(F(w, \tau) = \tau' \rightarrow \tau' \models \varphi)$
(2.5.4) $\tau \models \Box\varphi \equiv ((\forall \tau')((\exists w)F(w, \tau) = \tau' \rightarrow \tau' \models \varphi)$[4]

[4] The following expression is valid in predicate logic:

$(\forall x)(A(x) \rightarrow B) \equiv ((\exists x)A(x) \rightarrow B)$

and hence 2.5.3 and 2.5.4 are equivalent.

The antecedent of the right-hand expression in (2.5.4) is $(\exists w)F(w, \tau) = \tau'$. This stands for a relation, call it R, between τ' and τ. Relation $R(\tau', \tau)$ holds iff τ' is adequately reflected in τ as some possible world w. Indeed, in this case a sentence φ is true in τ' iff there is some w such that at τ it is true that φ is the case at w. By (2.5.4) we can thus replace (1.6.5) by:

1.6.5.1. $\tau \models \Box\varphi$ iff $\tau \models \varphi$ for all τ' such that $R(\tau', \tau)$
to provide a standard definition of the necessity operator $\Box$, and take (1.6.5) as a theorem. Then in the definition of $\tau \models$ expression $\Box\varphi$ is not given a greater complexity.

The appeal to the accessibility relation R on the set Π of worlds has an important consequence: we can define the logic of the operator $\Box$ by defining certain properties of R. In normal modal logic the accessibility relation R can be chosen freely, while in our case it is determined by the function F (but F can be chosen freely!). That is why the properties we may want to attach to R are ultimately related to properties of F. A first example is presented in the next theorem:

2.6. (*Non-existence of a maximal element*) $(\forall\tau)(\exists\tau')R(\tau, \tau')$.

Proof. By (2.2) we have: $\Box\varphi \rightarrow w\varphi$ and also $\Box\neg\varphi \rightarrow w\neg\varphi$, hence $(\Box\varphi \wedge \Box\neg\varphi) \rightarrow (w\varphi \wedge w\neg\varphi)$. Since, by (1.1) and (1.5.1) the consequent of this expression is a contradiction, we get:

2.7. $\Box\varphi \rightarrow \neg\Box\neg\varphi$.

A standard result is that (2.7) is defined by R's property of having no maximal element[5]. The property of R expressed by 2.6 can be traced back to the properties of function F. Since $R(\tau, \tau')$ is short for $(\exists w)F(w, \tau) = \tau'$, that R has no maximal element comes to: $(\forall\tau)(\exists\tau')(\exists w)F(w, \tau) = \tau'$. But this is simply a consequence of our assumption that F is a function.

[5] A nice proof appeals to the substitution method (van Benthem: 2001). The second order translation of (2.7) is

$$(\forall P)(\forall\tau)((\forall\tau')(R(\tau, \tau') \rightarrow P(\tau')) \rightarrow (\exists\tau')(R(\tau, \tau') \wedge P(\tau')))$$

Let us substitute $R(*, \tau')$ for $P(*)$. We get:

$$(\forall\tau)((\forall\tau')(R(\tau, \tau') \rightarrow R(\tau, \tau')) \rightarrow (\exists\tau')(R(\tau, \tau') \wedge R(\tau, \tau')))$$

i.e. exactly (2.6).

This section ends with the completeness theorem for BWM:

2.8. $\vdash_{\mathrm{BWM}} \varphi$ iff $\models_{\mathrm{BWM}} \varphi$.

To prove sufficiency we have to show that (2.1)–(2.4) are true in M. Let us take (2.3) as an example. Suppose that $\tau \models \varphi$ holds for every τ. Then, out of (1.3) we get $\tau \models w\varphi$ for every w, and hence $\tau \models \Box\varphi$ from (1.6.5). Suppose, conversely, that φ is not BWM-provable. Then we can show that there is a model M of BWM such that φ is not true in M. The procedure is analogous to the one used in the previous section to prove the completeness of BW. Thus, in $\mathsf{M} = \langle \Pi, \models, \mathcal{F} \rangle$, Π is the set of all BWM-maximal and consistent sets Σ of sentences of $\mathcal{I}'$; $\mathcal{F}$ is defined by: $\mathcal{F}(w, \Sigma) = \Sigma_w$; and put $\Sigma \models \varphi$ iff $\varphi \in \Sigma$, for every $\varphi \in S$. What we need is to examine sentences of the form $\Box\varphi$. We have to prove that:

2.9.1. If $\Sigma \models \Box\varphi$, then $\Box\varphi \in \Sigma$.
2.9.2. If $\Box\varphi \in \Sigma$, then $\Sigma \models \Box\varphi$.

In the case of (2.9.1), observe that $\Sigma \models \Box\varphi$ iff $\Sigma \models w\varphi$ for all w (by 1.6.5), iff $w\varphi \in \Sigma$ for every w (by induction); then $\Box\varphi \in \Sigma$ (by rule (2.4)). As for (2.9.2), we have: from $\Box\varphi \in \Sigma$ infer (by (2.2)) that $w\varphi \in \Sigma$ for every w; then $\Sigma \models w\varphi$ for every w (by induction), whence $\Sigma \models \Box\varphi$, by definition (1.6.5).

But, if φ is not BWM-provable, then the set $\{\neg\varphi\}$ is consistent and hence can be extended to a BWM-maximal and consistent set Σ of sentences of $\mathcal{I}'$; since $\neg\varphi \in \Sigma$, it follows that $\Sigma \models \neg\varphi$.

3. *Mirror Models*

As I mentioned above, our two sets of worlds — W and Π — must be kept distinct. The members of W were introduced syntactically, as elements of our formal language $\mathcal{I}$. The members of Π, however, were introduced semantically, as entities at which the sentences of our language are to be evaluated. Is there any connection between the two sets? In this section I will try to offer an answer to this question.

For each cardinal k, say that a modal logic L of possible worlds has the k property if the following holds: $\vdash_{\mathsf{L}} \varphi$ iff φ is true in all its models $\mathsf{M} = \langle \Pi, \models, \mathcal{F} \rangle$ for which $|\ \Pi\ | = k$. Now let $|\ W\ | = \mathsf{w}$. It is interesting to investigate the cases in which L has the w property, i.e. when $|\ \Pi\ | = \mathsf{w} = |\ W\ |$. In these cases a sentence φ is L-provable if it is true in all the models of L

in which the members of the two sets of worlds are one-to-one correlated[6]. The significance of a logic L's having the w property is that we can identify the two sets of worlds, and instead of working with both w and τ we can simply appeal to, say, w-type worlds. For example, we can then write $\mathcal{F}(w, w) = w$, meaning the each world adequately reflects itself. The logical and philosophical implications of this result are extensively discussed in my paper Miroiu (1999). Formally, the result requires the appeal to 'mirror models'. In this paper I will start with the canonical model of a modal logic L of possible worlds and will define these structures by using the mirroring procedure described in lemma (1.8).

The modal logic of possible worlds I will study below, call it BWM1, results by adding to BWM two new axioms (3.1) and (3.2), and a new rule (3.3). The main result of this section is the proof that BWM1 has the w property.

3.1. $\Box\varphi \rightarrow \Box\Box\varphi$
3.2. $\Box\Box\varphi \rightarrow \Box\varphi$
3.3. If $\vdash \Box\varphi$, then $\vdash \varphi$.

Usual calculations show that these expressions define important properties of the relation R:

3.1.1. (*transitivity*) If $R(\tau, \tau'))$ and $R(\tau', \tau'')$, then $R(\tau, \tau'')$.
3.2.1. (*density*) If $R(\tau, \tau')$, then there is some τ'' such that $R(\tau, \tau'')$ and $R(\tau'', \tau')$.
3.3.1. (*non-existence of a minimal element*) $(\forall\tau)(\exists\tau')R(\tau', \tau)$.

Consequently, at BWM1 relation R is transitive, dense, and (also given (2.6)) without both minimal and maximal elements. I shall first prove two lemmas. For every model M of BWM1 and every world in Π, it holds that:

3.4. For every w and w' there is some w'' such that for all φ, $\tau \models ww'\varphi$ iff $\tau \models w''\varphi$.

Proof. By (3.1) relation R is transitive: if there is some w_1 such that $\mathcal{F}(w_1, \tau) = \tau'$ and there is some w_2 such that $\mathcal{F}(w_2, \tau') = \tau''$, then there is some w_3 such that $\mathcal{F}(w_3, \tau) = \tau''$. Suppose that for the worlds w_1 and w_2 we have $\mathcal{F}(w_1, \tau) = \tau'$ and $\mathcal{F}(w_2, \tau') = \tau''$. Then for every sentence φ it holds that: $\tau \models w_1\varphi$ iff $\tau' \models \varphi$, and $\tau' \models w_2\varphi$ iff $\tau'' \models \varphi$. Given the

[6] If w is finite, i.e. the number of world symbols in our language $\mathcal{I}'$ is finite, then if a logic L has the w property, it must also have the finite model property.

definition of $\models$ for sentences of the form $w\psi$, it also holds that: $\tau \models w_1 w_2 \varphi$ iff $\tau'' \models n$. Since R is transitive, there should be some w_3 such that for all φ, $\tau \models w_3\varphi$ iff $\tau'' \models \varphi$. Thus for all φ, $\tau \models w_1 w_2 \varphi$ iff $\tau \models w_3\varphi$. Or, to put it differently, for every w_1 and w_2 there is some w_3 such that for all φ, $\tau \models w_1 w_2 \varphi$ iff $\tau \models w_3\varphi$.

3.5. For every w there are w' and w'' such that for all φ, $\tau \models w\varphi$ iff $\tau \models w'w''\varphi$.

Proof. By (3.2) R is dense: if $R(\tau, \tau')$, then there is some τ'' such that $R(\tau, \tau'')$ and $R(\tau'', \tau')$. Now, let w be a world in W and τ a world in Π. Then there is some τ' such that $F(w, \tau) = \tau'$, i.e. we have $R(\tau, \tau')$. Given the definition of $\models$ we have that for all φ, $\tau \models w\varphi$ iff $\tau' \models \varphi$. Since R is dense, there are some τ'', w' and w'' such that $F(w', \tau) = \tau''$ and $F(w'', \tau'') = \tau'$. Then for all φ, $\tau \models w'\varphi$ iff $\tau'' \models \varphi$, and $\tau'' \models w''\varphi$ iff $\tau' \models \varphi$. Hence for all φ, $\tau \models w'w''\varphi$ iff $\tau' \models n$. From this and: $\tau \models w\varphi$ iff $\tau' \models \varphi$, we get that: $\tau \models w'w''\varphi$ iff $\tau \models w\varphi$.

Now let $\Gamma = \langle \Pi, \models, F \rangle$ be the canonical model of BWM1. Γ is defined as follows: a) Π is the set of all BWM1-maximal consistent sets Σ of sentences; b) if φ is a sentence in S, then $\Sigma \models \varphi$ iff $\varphi \in \Sigma$; c) $F(w, \Sigma) = \Sigma'$ iff $\{\varphi : w\varphi \in \Sigma\} = \Sigma'$ (the fact that Σ' is a BWM1-maximal consistent set of sentences is a consequence of the mirroring lemma (1.8)). Relation R can be defined in the usual way in terms of F, by: $R(\Sigma, \Sigma')$ iff $(\exists w)(F(w, \Sigma) = \Sigma')$. To prove that Γ is the canonical model of BWM1 it is necessary to show that φ is a theorem of BWM1 iff $\Sigma \models \varphi$ for every Σ in Γ. The most important step is to prove that $\Sigma \models \varphi$ iff $\varphi \in \Sigma$ for all sentences. Obviously, the most difficult cases are $\varphi = w\psi$ and $\varphi = \Box\psi$. First, $\Sigma \models w\psi$ iff $F(w, \Sigma) \models \psi$, iff $\psi \in F(w, \Sigma)$, iff (by the definition of F) $w\psi \in \Sigma$. Secondly, we have: $\Sigma \models \Box\psi$ iff for every Σ', if $R(\Sigma, \Sigma')$, then $\Sigma' \models \psi$; iff for every Σ', if $(\exists w)(F(w, \Sigma) = \Sigma')$, then $\psi \in \Sigma'$; iff for every Σ' and w, if $F(w, \Sigma) = \Sigma'$), then $\psi \in \Sigma'$; iff for every w, $w\psi \in \Sigma$. Let us show that $w\psi \in \Sigma$ for every w iff $\Box\psi \in \Sigma$. The necessity part is directly entailed by (2.2); sufficiency yields by means of (2.4): for the set $\{\neg\Box\psi, w\psi : w \in W\}$ is inconsistent, and hence it is not included in Σ; hence we must have $\Box\psi \in \Sigma$.

Now we arrive at a procedure to bring about a mirror model Γ_Σ starting with a set Σ in Γ. For every Σ in Γ, the structure $\Gamma_\Sigma = \langle \Pi(\Sigma), \models, F \rangle$ is generated in the following way: a) $\Pi(\Sigma)$ is the set of all BWM1-maximal consistent sets $\Delta(w) = \{\psi : w\psi \in \Sigma\}$; b) if φ is a sentence in Σ, then $\Delta(w) \models \varphi$ iff $\varphi \in \Delta(w)$; c) if $\Delta \in \Pi(\Sigma)$ and $\Delta' \in \Pi(\Sigma)$, then $F(w, \Delta) = \Delta'$ iff $\{\varphi : w\varphi \in \Delta\} = \Delta'$.

I shall first prove that each structure Γ_Σ is indeed a model of BWM1 at which the sets W and $\Pi(\Sigma)$ of 'possible worlds' are one-to-one correlated.

3.6. For every Σ in Γ, $\Gamma_\Sigma = \langle \Pi(\Sigma), \models, F \rangle$ is a model of BWM1 and $|\ \Pi(\Sigma)\ | = \mathsf{w}$.

Proof. Obviously, at $\Gamma_\Sigma = \langle \Pi(\Sigma), \models, F \rangle$ each world in W is uniquely correlated with the 'world' $\Delta(w)$ in $\Pi(\Sigma)$, and hence $|\ \Pi(\Sigma)\ | = \mathsf{w}$.

First we need to show that function F is defined for every w and Σ in Γ_Σ. Suppose, indeed, that $\Delta' = F(w, \Delta)$, with $\Delta \in \Pi(\Sigma)$. Then, for every ψ, $\psi \in \Delta'$ iff $w\psi \in \Delta$. But, according to the definition of $\Delta \in \Pi(\Sigma)$, there is some w' such that $w'w\psi \in \Sigma$. But, since (3.1) is a theorem of BWM1, lemma (3.4) applies and hence there is some w'' in W such that for every ψ, $w''\psi \in \Sigma$ iff $w'w\psi \in \Sigma$. Thus, $\Delta' = \{\psi : w''\psi \in \Sigma\}$ must also be a member of the set $\Pi(\Sigma)$.[7]

Secondly, Γ_Σ satisfies all the axioms and rules of BWM1. I will shortly discuss only the rule (3.3): if $\vdash \Box\varphi$, then $\vdash \varphi$, because it causes some trouble. Suppose that for all sets $\Delta(w)$ in $\Pi(\Sigma)$ we have $\Box\varphi \in \Delta(w)$, but there is some $\Delta''(w'')$ such that $\neg\varphi \in \Delta''(w'')$. Then for every w, $\Sigma \models w\varphi$. We have thus $\Sigma \models \Box\varphi$, and by (3.2) we get $\Sigma \models \Box\Box\varphi$. But this is equivalent with: for every w and w', $\Sigma \models ww'\varphi$. Indeed, we have: $\Sigma \models \Box\Box\varphi$ iff for every w, $\Sigma \models w\Box\varphi$; iff for every w and Σ', if $F(w, \Sigma) = \Sigma'$, then $\Sigma' \models \Box\varphi$; iff for every w and Σ', if $F(w, \Sigma) = \Sigma'$, then for every w', $\Sigma' \models w'\varphi$; iff for every w and w', $F(w, \Sigma) \models w'\varphi$; iff for every w and w', $\Sigma \models ww'\varphi$. On the other hand, from $\Delta''(w'') \models \neg\varphi$ we get: $\Sigma \models w''\neg\varphi$. Hence for every w and w', it is the case that $\Sigma \models ww'\varphi$, but $\Sigma \models w''\neg\varphi$, for some w''. But no contradiction follows, because we cannot appeal to (3.5); for nothing guarantees that w'' itself is one of the for worlds for which it holds that: $\Sigma \models w''\varphi$ iff $\Sigma \models ww'\varphi$, for some pair w and w' of possible worlds.

It is possible, though, to appeal to the axiom (3.2). We already know that it is essential in the proof of lemma (3.5): for every w there are some w' and w'' such that for all φ, $\Sigma \models w\varphi$ iff $\Sigma \models w'w''\varphi$. What is its counterpart in our mirror model Γ_Σ generated by Σ? It comes to: for every $\Delta(w)$ there are

[7] It is also necessary to prove that at $\mathsf{M} = \langle \Pi, \models, F \rangle$ F is a function, i.e. that for every w in W and $\Delta(w')$ in Π, $F(w, \Delta(w'))$ picks up exactly one (maximally consistent) set of sentences. By the mirroring lemma, $F(w, \Delta(w'))$ is a BWM1-maximal consistent set of sentences. But we must be sure that it is also a member of the set Π, i.e. that it has the form $\Delta(w'')$, for some w''. Observe that $F(w, \Delta(w'))$ is the set of all those sentences φ such that $w\varphi$ is in $\Delta(w')$; thus, since $\mathsf{M} = \langle \Pi, \models, F \rangle$ is generated by the BWM1-maximal consistent set Σ, $F(w, \Delta(w'))$ is the set of all those sentences in Σ of the form $w'w\varphi$. By (3.3), there is some w'' such that $F(w, \Delta(w'))$ is the set of all those sentences φ with the property that $w''\varphi$ is in Σ; by the definition of M, $F(w, \Delta(w'))$ is exactly $\Delta(w'')$.

some w' and $\Delta'(w')$ such that $\Delta(w) \models \varphi$ iff $\Delta'(w') \models w''\varphi$, which in turn is equivalent with: for every $\Delta(w)$ there are $\Delta'(w')$ and w' and w'' such that $\mathcal{F}(w'', \Delta'(w')) = \Delta(w)$. At Γ_Σ this condition amounts to: for every $\Delta(w)$ there is some $\Delta'(w')$ such that $R(\Delta'(w'), \Delta(w))$. But as we already saw, the condition that there is no minimal element is expressed by (3.3). Hence this rule holds at our mirror model.

In order to prove that BWM1 has the w property, the crucial step is to show that:

3.7. $\{\varphi\}$ is BWM1-consistent iff there is some mirror model $\Gamma_\Sigma = \langle \Pi(\Sigma), \models, \mathcal{F} \rangle$ of BWM1 such that φ is true at some world in $\Pi(\Sigma)$.

Proof. If $\{\varphi\}$ is not BWM1-consistent, then there is no BWM1-maximal consistent set Σ of sets in the canonical model with $\varphi \in \Sigma$. But all the members of sets $\Pi(\Sigma)$ in every mirror model are BWM1-maximal consistent sets of sentences, and hence φ is not true in any world in $\Pi(\Sigma)$. Conversely, suppose that $\{\varphi\}$ is BWM1-consistent. Then, since Γ is the canonical model of BWM1, there is some BWM1-maximal consistent set Σ of sentences, such that $\varphi \in \Sigma$. What we still need is to show that φ is true at some world in $\Pi(\Sigma)$. By (3.3), there is some Σ' such that $R(\Sigma', \Sigma)$, i.e. for some w, $\mathcal{F}(w, \Sigma') = \Sigma$. Since $\varphi \in \Sigma$, we have $w\varphi \in \Sigma'$. But then we can generate a mirror model $\Gamma_{\Sigma'}$, and of course $\Sigma'(w) = \Sigma \in \Pi(\Sigma')$. Consequently, φ is true at some Π-world Σ in some mirror model.

Finally, we can state the theorem that a sentence is BWM1-provable if it is true in all the models in which the two sets Π and W of worlds are one-to-one correlated:

3.8. BWM1 has the w property.

Proof. Since Γ is the canonical model of BWM1, a sentence φ is true at Γ iff $\{\varphi\}$ is consistent. Then by (3.7) φ is true at Γ iff it is true in all the mirror models. Hence $\vdash_{\text{BWM1}} \varphi$ iff φ is true in all its models $\mathsf{M} = \langle \Pi, \models, \mathcal{F} \rangle$ at which $|\ \Pi\ | = \mathsf{w}$, which simply means that BWM1 has the w property.[8]

[8] A slightly stronger logic BWM2 with the w property can be obtained by adding to BWM (3.1) and the T-axiom:

3.9. $\Box\varphi \rightarrow \varphi$

One can easily check that (3.9) has both (3.2) and (3.3) as consequences. A standard result in modal semantics is that this axiom defines the property of reflexivity of relation R. At the canonical model, it comes to: for every Σ, $R(\Sigma, \Sigma)$. And this is in turn equivalent with: there is some world w in W such that for all φ, it holds that: $\varphi \in \Sigma$ iff $w\varphi \in \Sigma$. Consequently, at the mirror model generated by Σ, we have $\Delta(w) = \Sigma \in \Pi$.

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